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Oscillations and Nonoscillations of Half-Linear Difference Equations Generated by Deviating Arguments

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Abstract—For the half-linear difference equation

$$\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right]=\sum_{i=1}^{n}p_i(k)|y(g_i(k))|^{\alpha-1}y(g_i(k)),\qquad k\ge a,$$

where $\alpha>0$, we shall offer sufficient conditions for the oscillation of all solutions, as well as necessary and sufficient conditions for the existence of both bounded and unbounded nonoscillatory solutions. Several examples which dwell upon the importance of our results are also included. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Oscillatory solutions, Deviating arguments, Half-linear difference equations.

1. INTRODUCTION

In this paper, we shall consider the half-linear difference equation

$$\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right]=\sum_{i=1}^{n}p_i(k)|y(g_i(k))|^{\alpha-1}y(g_i(k)),\qquad k\ge a,\tag{1.1}$$

where $\alpha>0$ and Δ is the forward difference operator defined as $\Delta y(k)=y(k+1)-y(k)$. Further, for each $1\le i\le n$ we assume that

(I) $p_i(k)\ge 0$, $\max_{k\ge T}p_i(k)>0$ for any $T\ge a$, and
(II) $g_i:Z^+\cup\{0\}\to Z$ is such that $\Delta g_i(k)>0$ eventually and $\lim_{k\to\infty}g_i(k)=\infty$.

By a solution of (1.1), we mean a nontrivial sequence $\{y(k)\}$ satisfying (1.1) for $k\ge a$. A solution $\{y(k)\}$ is said to be oscillatory if it is neither eventually positive nor negative, and nonoscillatory otherwise.

The literature on the oscillation criteria of difference equations is vast, e.g., see [1,2] which cover a large number of recent papers. In particular, we refer to [3,4] where oscillations of equations similar to (1.1), but employing techniques different from the present paper have been discussed. Other related work can also be found in [5–10]. We note that an equation related to the continuous version of (1.1)

$$\left[p(t)|y'|^{\alpha-1}y'\right]'+q(t)|y|^{\alpha-1}y=0,\qquad t\ge c,\tag{1.2}$$

where $p(t),q(t)>0$ has been the subject matter of many recent investigations, e.g., see [11–19]. The oscillatory results for (1.2) find application to quasilinear degenerate elliptic partial differential equations. Further, Kusano and Lalli [20] have recently considered the continuous analog

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX of (1.1)

$$(|y'|^{\alpha-1}y')' = \sum_{i=1}^{n} p_i(t)|y(g_i(t))|^{\alpha-1}y(g_i(t)), \qquad t \ge c. \tag{1.3}$$

For other works on differential equations with deviating arguments, we refer to [21–26]. Our results not only extend the known theorems for (1.2),(1.3) to a discrete case, but also include several other known criteria discussed in [1].

The plan of the paper is as follows: in Section 2, we shall provide sufficient conditions for the oscillation of all bounded as well as unbounded solutions of equation (1.1). As a consequence, sufficient conditions for the oscillation of all solutions of (1.1) are obtained. In Section 3, we shall study the nonoscillatory behavior of (1.1), and establish necessary and sufficient conditions for the existence of both bounded and unbounded nonoscillatory solutions.

2. OSCILLATION THEOREMS FOR (1.1)

We begin by considering the following difference inequality

$$\left\{\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] - p(k)|y(g(k))|^{\alpha-1}y\left(g(k)\right)\right\} \operatorname{sgn} y\left(g(k)\right) \ge 0, \qquad k \ge a, \tag{2.1}$$

where $\alpha > 0$. Further, it is assumed that

(I) $p(k) \ge 0$, $\max_{k\ge T} p(k) > 0$ for any $T \ge a$, and
(II) $g : Z^+ \cup \{0\} \to Z$ is such that $\Delta g(k) > 0$ eventually and $\lim_{k\to\infty} g(k) = \infty$.

Let $\{y(k)\}$ be a nonoscillatory solution of (2.1). Then, it is clear from (2.1) that $\Delta y(k)$ is eventually of fixed sign. Hence, depending on whether the nonoscillatory solution $\{y(k)\}$ is bounded or unbounded, we have for sufficiently large K,

$$y(k)\Delta y(k) < 0 \quad \text{or} \quad y(k)\Delta y(k) > 0, \qquad k \ge K. \tag{2.2}$$

With no loss of generality, let the nonoscillatory solution $\{y(k)\}$ be such that $y(k) > 0$ for $k \ge K$. Then, (2.1) reduces to

$$\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] \ge p(k)|y(g(k))|^{\alpha-1}y(g(k)) \ge 0, \qquad k \ge K_1, \tag{2.3}$$

where K_1 $(> K)$ satisfies

$$\min_{k\ge K_1} g(k) \ge K. \tag{2.4}$$

It follows from (2.3) that for $k \ge K_1$,

$$|\Delta y(k)|^{\alpha-1}\Delta y(k) = |\Delta y(k)|^{\alpha} \operatorname{sgn}\{\Delta y(k)\}$$

is nondecreasing, i.e., for $k_2 \ge k_1 \ge K_1$,

$$|\Delta y(k_2)|^{\alpha} \operatorname{sgn}\{\Delta y(k_2)\} \ge |\Delta y(k_1)|^{\alpha} \operatorname{sgn}\{\Delta y(k_1)\},$$

which, together with the fact that $\Delta y(k)$ is eventually of fixed sign, leads to $\Delta y(k)$ is nondecreasing for $k \ge K_1$. Hence, $y(k)$ is convex for $k \ge K_1$.

THEOREM 2.1. *Suppose that $g(k)$ is a retarded argument such that $g(k) < k$ for $k \ge a$, and one of the following holds:*

$$\limsup_{k\to\infty} \sum_{\ell=g(k)}^{k-1} p(\ell)[g(k) + 1 - g(\ell)]^{\alpha} > 1, \tag{2.5}$$

$$\limsup_{k\to\infty} \sum_{\ell=g(k)}^{k} \left[\sum_{\tau=\ell}^{k} p(\tau)\right]^{1/\alpha} > 1. \tag{2.6}$$

Then, all bounded solutions of (2.1) are oscillatory.

PROOF. Let $\{y(k)\}$ be a nonoscillatory bounded solution of (2.1), say, $y(k) > 0$ for $k \geq K \geq a$. We shall consider only this case because the proof for the case $y(k) < 0$ for $k \geq K \geq a$ is similar. By (2.2),

$$\Delta y(k) < 0, \qquad k \geq K_1, \tag{2.7}$$

where K_1 is defined in (2.4).

First, suppose that (2.5) holds. We have seen earlier that $y(k)$ is convex for $k \geq K_1$. Hence,

$$y(\sigma) \geq y(\tau+1) - \Delta y(\tau) \cdot (\tau + 1 - \sigma) \geq -\Delta y(\tau) \cdot (\tau + 1 - \sigma), \qquad \tau + 1 \geq \sigma \geq K_1. \tag{2.8}$$

Let $\tau = g(k)$ and $\sigma = g(\ell)$ in (2.8) to get

$$y(g(\ell)) \geq -\Delta y(g(k)) \cdot [g(k) + 1 - g(\ell)], \qquad k \geq \ell \geq K_2, \tag{2.9}$$

where K_2 ($> K_1$) satisfies

$$\min_{k \geq K_2} g(k) \geq K_1. \tag{2.10}$$

In view of (2.7), it follows from (2.9) that

$$p(\ell)[y(g(\ell))]^\alpha \geq p(\ell)[-\Delta y(g(k))]^\alpha [g(k) + 1 - g(\ell)]^\alpha, \qquad k \geq \ell \geq K_2,$$

which on using (2.1) provides

$$\begin{aligned} p(\ell)[-\Delta y(g(k))]^\alpha [g(k) + 1 - g(\ell)]^\alpha &\leq \Delta\left[|\Delta y(\ell)|^{\alpha-1}\Delta y(\ell)\right] \\ &= \Delta\left[-|\Delta y(\ell)|^\alpha\right] \\ &= -\Delta\left[(-\Delta y(\ell))^\alpha\right], \qquad k \geq \ell \geq K_2. \end{aligned} \tag{2.11}$$

Summing (2.11) from $g(k)$ to $(k-1)$ (noting that $g(k) < k$), we get

$$\begin{aligned} &[-\Delta y(g(k))]^\alpha \sum_{\ell=g(k)}^{k-1} p(\ell)[g(k) + 1 - g(\ell)]^\alpha - [-\Delta y(g(k))]^\alpha \\ &\qquad \leq -[-\Delta y(k)]^\alpha \leq 0, \qquad k \geq K_3, \end{aligned} \tag{2.12}$$

where K_3 ($> K_2$) satisfies

$$\min_{k \geq K_3} g(k) \geq K_2. \tag{2.13}$$

From (2.12), we find

$$[-\Delta y\,(g(k))]^\alpha \left\{ \sum_{\ell=g(k)}^{k-1} p(\ell)\,[g(k) + 1 - g(\ell)]^\alpha - 1 \right\} \leq 0, \qquad k \geq K_3,$$

which is a contradiction to (2.5).

Next, suppose that (2.6) holds. We sum (2.1) from σ to $(k-1)$ to obtain

$$-[-\Delta y(k)]^\alpha + [-\Delta y(\sigma)]^\alpha \geq \sum_{\ell=\sigma}^{k-1} p(\ell)\,[y(g(\ell))]^\alpha, \qquad k \geq \sigma + 1 \geq K_1 + 1. \tag{2.14}$$

In view of (2.7), it follows from (2.14) that

$$-\Delta y(\sigma) \geq \left\{ \sum_{\ell=\sigma}^{k-1} p(\ell)[y(g(\ell))]^\alpha \right\}^{1/\alpha}, \qquad k \geq \sigma + 1 \geq K_1 + 1. \tag{2.15}$$

Now, we write

$$y(\ell) = y(k) + \sum_{\sigma=\ell}^{k-1} [-\Delta y(\sigma)], \qquad k \ge \ell + 1 \ge K_1 + 1,$$

which on using (2.15) yields

$$\begin{aligned} y(\ell) &\ge y(k) + \sum_{\sigma=\ell}^{k-1} \left\{ \sum_{\tau=\sigma}^{k-1} p(\tau)[y(g(\tau))]^{\alpha} \right\}^{1/\alpha} \\ &\ge \sum_{\sigma=\ell}^{k-1} \left\{ \sum_{\tau=\sigma}^{k-1} p(\tau)[y(g(\tau))]^{\alpha} \right\}^{1/\alpha}, \qquad k \ge \ell + 1 \ge K_1 + 1. \end{aligned} \tag{2.16}$$

Since $g(k-1) < k-1$, in (2.16) we may substitute $\ell = g(k-1)$ to get

$$\begin{aligned} y(g(k-1)) &\ge \sum_{\sigma=g(k-1)}^{k-1} \left\{ \sum_{\tau=\sigma}^{k-1} p(\tau)[y(g(\tau))]^{\alpha} \right\}^{1/\alpha} \\ &\ge \sum_{\sigma=g(k-1)}^{k-1} \left\{ \sum_{\tau=\sigma}^{k-1} p(\tau)[y(g(k-1))]^{\alpha} \right\}^{1/\alpha} \\ &= y(g(k-1)) \sum_{\sigma=g(k-1)}^{k-1} \left[\sum_{\tau=\sigma}^{k-1} p(\tau) \right]^{1/\alpha}, \qquad k \ge K_2 + 1. \end{aligned}$$

The above inequality is equivalent to

$$y(g(k-1)) \left\{ 1 - \sum_{\sigma=g(k-1)}^{k-1} \left[\sum_{\tau=\sigma}^{k-1} p(\tau) \right]^{1/\alpha} \right\} \ge 0, \qquad k \ge K_2 + 1. \tag{2.17}$$

Inequality (2.17) is a contradiction to (2.6). This completes the proof of the theorem.

THEOREM 2.2. *Suppose that $g(k)$ is an advanced argument such that one of the following holds:*

(a) *$g(k) > k$, for $k \ge a$ and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g(k)-1} p(\ell)[g(\ell) - g(k)]^{\alpha} > 1, \tag{2.18}$$

(b) *$g(k) > k + 1$, for $k \ge a$ and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g(k)-1} \left[\sum_{\tau=k}^{\ell-1} p(\tau) \right]^{1/\alpha} > 1. \tag{2.19}$$

Then, all unbounded solutions of (2.1) are oscillatory.

PROOF. Let $\{y(k)\}$ be a nonoscillatory unbounded solution of (2.1), say, $y(k) > 0$ for $k \ge K \ge a$. Then, from (2.2) we have

$$\Delta y(k) > 0, \qquad k \ge K_1, \tag{2.20}$$

where K_1 is defined in (2.4).

(a) We suppose that (2.18) holds. Since $y(k)$ is convex for $k \ge K_1$, we find

$$y(\sigma) \ge y(\tau) + \Delta y(\tau) \cdot (\sigma - \tau) \ge \Delta y(\tau) \cdot (\sigma - \tau), \qquad \sigma \ge \tau \ge K_1. \tag{2.21}$$

Substituting $\sigma = g(\ell)$ and $\tau = g(k)$ in (2.21), we get

$$y(g(\ell)) \geq \Delta y(g(k)) \cdot [g(\ell) - g(k)], \qquad \ell \geq k \geq K_2, \tag{2.22}$$

where K_2 is defined in (2.10). In view of (2.20) and (2.1), it is immediate from (2.22) that

$$p(\ell)[\Delta y(g(k))]^\alpha [g(\ell) - g(k)]^\alpha \leq p(\ell)[y(g(\ell))]^\alpha \leq \Delta\left[(\Delta y(\ell))^\alpha\right], \qquad \ell \geq k \geq K_2. \tag{2.23}$$

Summing (2.23) from k to $[g(k) - 1]$ (noting that $g(k) > k$), we get

$$[\Delta y(g(k))]^\alpha - [\Delta y(k)]^\alpha \geq [\Delta y(g(k))]^\alpha \sum_{\ell=k}^{g(k)-1} p(\ell)[g(\ell) - g(k)]^\alpha, \qquad k \geq K_2,$$

which implies

$$[\Delta y(g(k))]^\alpha \left\{1 - \sum_{\ell=k}^{g(k)-1} p(\ell)[g(\ell) - g(k)]^\alpha\right\} \geq 0, \qquad k \geq K_2. \tag{2.24}$$

Inequality (2.24) is a contradiction to (2.18).

(b) Suppose that (2.19) holds. Summing (2.1) from k to $(\sigma - 1)$, we obtain

$$[\Delta y(\sigma)]^\alpha - [\Delta y(k)]^\alpha \geq \sum_{\ell=k}^{\sigma-1} p(\ell)[y(g(\ell))]^\alpha, \qquad \sigma \geq k+1 \geq K_1 + 1. \tag{2.25}$$

In view of (2.20), it is clear from (2.25) that

$$\Delta y(\sigma) \geq \left\{\sum_{\ell=k}^{\sigma-1} p(\ell)[y(g(\ell))]^\alpha\right\}^{1/\alpha}, \qquad \sigma \geq k+1 \geq K_1 + 1. \tag{2.26}$$

Writing

$$y(\ell) = y(k) + \sum_{\sigma=k}^{\ell-1} \Delta y(\sigma), \qquad \ell \geq k+1 \geq K_1 + 1,$$

a substitution of (2.26) provides

$$\begin{aligned} y(\ell) &\geq y(k) + \sum_{\sigma=k}^{\ell-1} \left\{\sum_{\tau=k}^{\sigma-1} p(\tau)[y(g(\tau))]^\alpha\right\}^{1/\alpha} \\ &\geq \sum_{\sigma=k}^{\ell-1} \left\{\sum_{\tau=k}^{\sigma-1} p(\tau)[y(g(\tau))]^\alpha\right\}^{1/\alpha}, \qquad \ell \geq k+1 \geq K_1 + 1. \end{aligned} \tag{2.27}$$

Since $g(k) > k + 1$ in (2.27), we may substitute $\ell = g(k)$ to get

$$\begin{aligned} y(g(k)) &\geq \sum_{\sigma=k}^{g(k)-1} \left\{\sum_{\tau=k}^{\sigma-1} p(\tau)[y(g(\tau))]^\alpha\right\}^{1/\alpha} \\ &\geq \sum_{\sigma=k}^{g(k)-1} \left\{\sum_{\tau=k}^{\sigma-1} p(\tau)[y(g(k))]^\alpha\right\}^{1/\alpha} \\ &= y(g(k)) \sum_{\sigma=k}^{g(k)-1} \left[\sum_{\tau=k}^{\sigma-1} p(\tau)\right]^{1/\alpha}, \qquad k \geq K_2 + 1. \end{aligned} \tag{2.28}$$

Inequality (2.28) is equivalent to

$$y(g(k)) \left\{1 - \sum_{\sigma=k}^{g(k)-1} \left[\sum_{\tau=k}^{\sigma-1} p(\tau)\right]^{1/\alpha}\right\} \geq 0, \qquad k \geq K_2 + 1,$$

and this contradicts (2.19). The proof of the theorem is now complete.

THEOREM 2.3.

(a) *Suppose that there exists a* i, $1 \le i \le n$ *such that* $g_i(k) < k$ *for* $k \ge a$, *and one of the following holds:*

$$\limsup_{k\to\infty} \sum_{\ell=g_i(k)}^{k-1} p_i(\ell)[g_i(k)+1-g_i(\ell)]^\alpha > 1, \tag{2.29}$$

$$\limsup_{k\to\infty} \sum_{\ell=g_i(k)}^{k} \left[\sum_{\tau=\ell}^{k} p_i(\tau)\right]^{1/\alpha} > 1. \tag{2.30}$$

Then, all bounded solutions of (1.1) are oscillatory.

(b) *Suppose that there exists a* j, $1 \le j \le n$ *such that one of the following holds:*

(i) $g_j(k) > k$, *for* $k \ge a$ *and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g_j(k)-1} p_j(\ell)[g_j(\ell)-g_j(k)]^\alpha > 1, \tag{2.31}$$

(ii) $g_j(k) > k+1$, *for* $k \ge a$ *and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g_j(k)-1} \left[\sum_{\tau=k}^{\ell-1} p_j(\tau)\right]^{1/\alpha} > 1. \tag{2.32}$$

Then, all unbounded solutions of (1.1) are oscillatory.

(c) *Suppose that there exist* i *and* j, $1 \le i, j \le n$ *such that* $g_i(k)$ *and* $g_j(k)$ *satisfy the conditions of (a) and (b), respectively. Then, all solutions of (1.1) are oscillatory.*

PROOF.

(a) Let $\{y(k)\}$ be a nonoscillatory bounded solution of (1.1). Then, from (1.1) we see that the following inequality holds for sufficiently large k:

$$\left\{\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] - p_i(k)|y(g_i(k))|^{\alpha-1}y(g_i(k))\right\} \operatorname{sgn} y(g_i(k)) \ge 0. \tag{2.33}$$

However, by Theorem 2.1 the inequality (2.33) cannot have any nonoscillatory bounded solutions.

(b) Let $\{y(k)\}$ be a nonoscillatory unbounded solution of (1.1). Then, it follows from (1.1) that the inequality (2.33) (with i replaced by j) holds for sufficiently large k. However, by Theorem 2.2 this is not possible because all unbounded solutions of (2.33) are oscillatory.

(c) This is obvious from (a) and (b).

EXAMPLE 2.1. *Consider the difference equations*

$$\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] = c|y(k-\sigma)|^{\alpha-1}y(k-\sigma), \qquad k \ge \sigma, \tag{2.34}$$

$$\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] = d|y(k+\sigma_1)|^{\alpha-1}y(k+\sigma_1), \qquad k \ge 0, \tag{2.35}$$

and

$$\begin{aligned}\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] &= c|y(k-\sigma)|^{\alpha-1}y(k-\sigma) \\ &\quad + d|y(k+\sigma_1)|^{\alpha-1}y(k+\sigma_1), \qquad k \ge \sigma,\end{aligned} \tag{2.36}$$

where c, d, α *are positive real numbers,* σ (≥ 1) *and* σ_1 (≥ 2) *are integers.*

It can easily be verified that conditions (2.29) and (2.30) are equivalent to

$$\sum_{\ell=2}^{\sigma+1} c\ell^\alpha > 1 \tag{2.37}$$

and

$$\sum_{\ell=1}^{\sigma+1}(c\ell)^{1/\alpha} > 1, \tag{2.38}$$

respectively. Thus, by Theorem 2.3(a), if (2.37) or (2.38) is satisfied, then all bounded solutions of equation (2.34) are oscillatory. This is particularly so when $\alpha = 2$, $\sigma = 4$, and $c > 0.0143$.

We also find that conditions (2.31) and (2.32) reduce to

$$\sum_{\ell=1}^{\sigma_1-1} d\ell^\alpha > 1 \tag{2.39}$$

and

$$\sum_{\ell=1}^{\sigma_1-1} (d\ell)^{1/\alpha} > 1, \tag{2.40}$$

respectively. It follows from Theorem 2.3(b) that if (2.39) or (2.40) are fulfilled, then all unbounded solutions of equation (2.35) are oscillatory. As a particular example, this is the case when $\alpha = 2$, $\sigma_1 = 3$, and $d > 0.172$.

By Theorem 2.3(c), all solutions of (2.36) are oscillatory provided one of (2.37) and (2.38), and one of (2.39) and (2.40) hold. For example, when

$$\alpha = 2, \qquad \sigma = 4, \qquad \sigma_1 = 3, \qquad c > 0.0143, \qquad d > 0.172,$$

all solutions of (2.36) are oscillatory. Indeed, for any c, d such that $c - d = 8$ ($c > 0.0143$, $d > 0.172$), one such solution is given by $\{y(k)\} = \{(-1)^k\}$.

3. NONOSCILLATION THEOREMS FOR (1.1)

Let $\{y(k)\}$ be a nonoscillatory solution of (1.1). Then, from (1.1) we see that $\Delta y(k)$ is eventually of fixed sign. Hence, depending on whether the nonoscillatory solution $\{y(k)\}$ is bounded or unbounded, we have for sufficiently large K,

$$y(k)\Delta y(k) < 0 \quad \text{or} \quad y(k)\Delta y(k) > 0, \qquad k \geq K. \tag{3.1}$$

THEOREM 3.1. *Equation (1.1) has a unbounded nonoscillatory solution $\{y(k)\}$ such that*

$$\lim_{k\to\infty} \frac{y(k)}{k} = \text{constant } \neq 0, \tag{3.2}$$

if and only if

$$\sum^{\infty} p_i(k)\,[g_i(k)]^\alpha < \infty, \qquad 1 \leq i \leq n. \tag{3.3}$$

PROOF. First, we suppose that equation (1.1) has a unbounded nonoscillatory solution $\{y(k)\}$ satisfying (3.2). With no loss of generality, let $y(k) > 0$ for $k \geq K \geq a$. It follows from (3.1) that $\Delta y(k) > 0$ for $k \geq K$. Further, (3.2) implies that $L \equiv \lim_{k\to\infty} \Delta y(k)$ (= constant) is finite. Now, we sum (1.1) from ℓ to ∞ to get

$$\Delta y(\ell) = \left\{ L^\alpha - \sum_{\tau=\ell}^{\infty}\sum_{i=1}^{n} p_i(\tau)[y(g_i(\tau))]^\alpha \right\}^{1/\alpha}, \qquad \ell \geq K_1, \tag{3.4}$$

where K_1 ($> K$) satisfies

$$\min_{1\leq i\leq n}\ \min_{k\geq K_1} g_i(k) \geq K. \tag{3.5}$$

Summing (3.4) again from K_1 to k provides

$$y(k) = y(K_1) + \sum_{\ell=K_1}^{k} \left\{ L^\alpha - \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau)[y(g_i(\tau))]^\alpha \right\}^{1/\alpha}. \tag{3.6}$$

It is clear from (3.6) that the following must hold:

$$\sum^{\infty} \sum_{i=1}^{n} p_i(\tau)[y(g_i(\tau))]^\alpha < \infty, \tag{3.7}$$

for otherwise, the right side of (3.6) tends to $-\infty$ as $k \to \infty$, contradicting the fact that $y(k)$ is eventually positive. Since, in view of (3.2), we have

$$\lim_{\tau \to \infty} \frac{y(g_i(\tau))}{g_i(\tau)} = \text{constant}, \qquad 1 \le i \le n,$$

condition (3.7) implies

$$\sum^{\infty} \sum_{i=1}^{n} p_i(\tau)[g_i(\tau)]^\alpha < \infty,$$

which is equivalent to (3.3).

Next, suppose that (3.3) holds. Let $N > 0$ be a fixed arbitrary number. Choose $K_2 \ge a$ so large that

$$K_3 = \min_{1 \le i \le n} \min_{k \ge K_2} g_i(k) \ge a, \tag{3.8}$$

and in view of (3.3),

$$\sum_{\tau=K_2+1}^{\infty} \sum_{i=1}^{n} p_i(\tau - 1)\, [g_i(\tau - 1)]^\alpha \le 1 - 4^{-\alpha}. \tag{3.9}$$

Let $\bar{K} = \min\{K_2, K_3\}$,

$$B \equiv \{y(k) : y(k) \text{ is defined for } k \ge \bar{K}\} \tag{3.10}$$

and

$$Y \equiv \left\{ y(k) \in B : \frac{N}{4}(k - K_2) \le y(k) \le N(k - K_2),\ k \ge K_2;\ y(k) = 0,\ \bar{K} \le k \le K_2 \right\}.$$

Define $G : Y \to B$ by

$$(Gy)(k) = \sum_{\ell=K_2}^{k-1} \left\{ N^\alpha - \sum_{\tau=\ell+1}^{\infty} \sum_{i=1}^{n} p_i(\tau - 1)[y(g_i(\tau - 1))]^\alpha \right\}^{1/\alpha}, \qquad k \ge \bar{K}.$$

Let $y(k) \in Y$. If $\bar{K} \le k \le K_2$, then it is clear that $(Gy)(k) = 0$. For $k \ge K_2$, we have

$$(Gy)(k) \le \sum_{\ell=K_2}^{k-1} (N^\alpha - 0)^{1/\alpha} = N\,(k - K_2),$$

and on using (3.9),

$$\begin{aligned}
(Gy)(k) &\ge \sum_{\ell=K_2}^{k-1} \left\{ N^\alpha - \sum_{\tau=\ell+1}^{\infty} \sum_{i=1}^{n} p_i(\tau - 1)[\max\{N(g_i(\tau - 1) - K_2), 0\}]^\alpha \right\}^{1/\alpha} \\
&\ge \sum_{\ell=K_2}^{k-1} \left\{ N^\alpha - \sum_{\tau=\ell+1}^{\infty} \sum_{i=1}^{n} p_i(\tau - 1)[N\, g_i(\tau - 1)]^\alpha \right\}^{1/\alpha} \\
&\ge \sum_{\ell=K_2}^{k-1} \left\{ N^\alpha - \sum_{\tau=K_2+1}^{\infty} \sum_{i=1}^{n} p_i(\tau - 1)[N\, g_i(\tau - 1)]^\alpha \right\}^{1/\alpha} \\
&\ge \sum_{\ell=K_2}^{k-1} \left[N^\alpha - N^\alpha(1 - 4^{-\alpha}) \right]^{1/\alpha} = \frac{N}{4}\,(k - K_2).
\end{aligned}$$

Hence, $G(Y) \subseteq Y$. It is clear that Y is a closed and compact subset of B and $G(Y)$ is relatively compact in B. Therefore, by Schauder fixed point theorem, G has a fixed point in Y given by

$$y(k) = \sum_{\ell=K_2}^{k-1} \left\{ N^\alpha - \sum_{\tau=\ell+1}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[y(g_i(\tau-1))]^\alpha \right\}^{1/\alpha}, \qquad k \geq \bar{K}. \tag{3.11}$$

It can easily be checked that this $y(k)$ satisfies (1.1). To see that (3.2) is also fulfilled, we note that

$$\begin{aligned} \lim_{k\to\infty} \frac{y(k)}{k} &= \lim_{k\to\infty} \Delta y(k) \\ &= \lim_{k\to\infty} \left\{ N^\alpha - \sum_{\tau=k+1}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[y(g_i(\tau-1))]^\alpha \right\}^{1/\alpha} = N. \end{aligned}$$

Thus, the $y(k)$ given in (3.11) is a unbounded nonoscillatory solution of (1.1) such that (3.2) holds. The proof of the theorem is complete.

THEOREM 3.2. *Equation (1.1) has a bounded nonoscillatory solution $\{y(k)\}$ such that*

$$\lim_{k\to\infty} y(k) = \text{ constant } \neq 0, \tag{3.12}$$

if and only if

$$\sum^{\infty} \left[\sum_{\ell=k}^{\infty} p_i(\ell) \right]^{1/\alpha} < \infty, \qquad 1 \leq i \leq n. \tag{3.13}$$

PROOF. First, we suppose that equation (1.1) has a bounded nonoscillatory solution $\{y(k)\}$ satisfying (3.12). Again, let $y(k) > 0$ for $k \geq K \geq a$. It follows from (3.1) that $\Delta y(k) < 0$ for $k \geq K$. Further, (3.12) implies that $\lim_{k\to\infty} \Delta y(k) = 0$, and $M \equiv \lim_{k\to\infty} y(k)$ is finite. Summing (1.1) from ℓ to ∞, we obtain

$$[-\Delta y(\ell)]^\alpha = \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau)\,[y(g_i(\tau))]^\alpha, \qquad \ell \geq K_1, \tag{3.14}$$

where K_1 is defined in (3.5). A second summation of (3.14) from k to ∞ provides

$$y(k) = M + \sum_{\ell=k}^{\infty} \left\{ \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau)[y(g_i(\tau))]^\alpha \right\}^{1/\alpha}, \qquad k \geq K_1. \tag{3.15}$$

In view of the fact that $y(k)$ is bounded, condition (3.13) readily follows from (3.15).

Next, suppose that (3.13) holds. Let $N > 0$ be a fixed arbitrary number. Choose $K_2 \geq a$ so large that (3.8) holds, and also, in view of (3.13),

$$\sum_{\ell=K_2+1}^{\infty} \left[\sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1) \right]^{1/\alpha} \leq \frac{3}{4}. \tag{3.16}$$

Let

$$Y \equiv \left\{ y(k) \in B : N \leq y(k) \leq 4N,\ k \geq \bar{K} \right\},$$

where B is given in (3.10). We define $G : Y \to B$ by

$$(Gy)(k) = \begin{cases} N + \sum_{\ell=k+1}^{\infty} \left\{ \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[y(g_i(\tau-1))]^\alpha \right\}^{1/\alpha}, & k \geq K_2, \\ (Gy)(K_2), & \bar{K} \leq k \leq K_2. \end{cases}$$

Let $y(k) \in Y$, for $k \geq \bar{K}$ obviously we have $(Gy)(k) \geq N$, and in view of (3.16),

$$\begin{aligned}(Gy)(k) &\leq N + \sum_{\ell=K_2+1}^{\infty} \left[\sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)(4N)^{\alpha} \right]^{1/\alpha} \\ &\leq N + 4N \cdot \frac{3}{4} = 4N.\end{aligned}$$

Thus, $G(Y) \subseteq Y$. It is clear that Y is a closed and compact subset of B and $G(Y)$ is relatively compact in B. Therefore, by Schauder fixed point theorem, G has a fixed point in Y given by

$$y(k) = N + \sum_{\ell=k+1}^{\infty} \left\{ \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[y(g_i(\tau-1))]^{\alpha} \right\}^{1/\alpha}, \qquad k \geq \bar{K}. \tag{3.17}$$

Clearly, this $y(k)$ satisfies (3.12). Further, since

$$\Delta y(k) = -\left\{ \sum_{\tau=k+1}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[y(g_i(\tau-1))]^{\alpha} \right\}^{1/\alpha} < 0,$$

we find

$$|\Delta y(k)|^{\alpha-1} \Delta y(k) = -[-\Delta y(k)]^{\alpha} = -\sum_{\tau=k+1}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[y(g_i(\tau-1))]^{\alpha},$$

which provides

$$\Delta\left[|\Delta y(k)|^{\alpha-1} \Delta y(k)\right] = \sum_{i=1}^{n} p_i(k)[y(g_i(k))]^{\alpha} = \sum_{i=1}^{n} p_i(k)|y(g_i(k))|^{\alpha-1} y(g_i(k)).$$

Hence, the $y(k)$ given in (3.17) is a bounded nonoscillatory solution of (1.1) such that (3.12) holds. This completes the proof of the theorem.

THEOREM 3.3. *Suppose that there exists a j, $1 \leq j \leq n$ such that*

$$g_j(k) < k \quad \text{and} \quad p_j(k) > 0, \qquad k \geq a. \tag{3.18}$$

Further, suppose that there exists a positive decreasing function $q(k)$ such that

$$q(k) \geq \sum_{\ell=k+1}^{\infty} \left\{ \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[q(g_i(\tau-1))]^{\alpha} \right\}^{1/\alpha}, \qquad k \geq K_0, \tag{3.19}$$

where K_0 ($> a$) satisfies

$$\min_{1 \leq i \leq n} \min_{k \geq K_0} g_i(k) \geq a.$$

Then, (1.1) has a decaying nonoscillatory solution $\{y(k)\}$ such that

$$\lim_{k \to \infty} y(k) = 0. \tag{3.20}$$

PROOF. Let

$$B \equiv \{y(k) : y(k) \text{ is defined for } k \geq K_0\}$$

and

$$Y \equiv \{y(k) \in B : 0 \leq y(k) \leq q(k),\ k \geq K_0\}.$$

For each $y(k) \in Y$, we define

$$\bar{y}(k) = \begin{cases} y(k), & k \geq K_0, \\ y(K_0) + q(k) - q(K_0), & a \leq k \leq K_0. \end{cases}$$

We note that for each $y(k) \in Y$,

$$\bar{y}(k) \leq q(k), \qquad k \geq a. \tag{3.21}$$

Define $G : Y \to B$ by

$$(Gy)(k) = \sum_{\ell=k+1}^{\infty} \left\{ \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1) \left[\bar{y}(g_i(\tau-1))\right]^{\alpha} \right\}^{1/\alpha}, \qquad k \geq K_0.$$

Clearly, for $y(k) \in Y$, $k \geq K_0$, we have $(Gy)(k) \geq 0$, and on using (3.21) and (3.19),

$$(Gy)(k) \leq \sum_{\ell=k+1}^{\infty} \left\{ \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[q(g_i(\tau-1))]^{\alpha} \right\}^{1/\alpha} \leq q(k).$$

Hence, $G(Y) \subseteq Y$. Since Y is a closed and compact subset of B and $G(Y)$ is relatively compact in B, it follows from Schauder fixed point theorem that G has a fixed point in Y given by

$$y(k) = \sum_{\ell=k+1}^{\infty} \left\{ \sum_{\tau=\ell}^{\infty} \sum_{i=1}^{n} p_i(\tau-1)[\bar{y}(g_i(\tau-1))]^{\alpha} \right\}^{1/\alpha}, \qquad k \geq K_0. \tag{3.22}$$

Relation (3.22) readily yields

$$\Delta y(k) = -\left\{ \sum_{\tau=k+1}^{\infty} \sum_{i=1}^{n} p_i(\tau-1) \left[\bar{y}(g_i(\tau-1))\right]^{\alpha} \right\}^{1/\alpha},$$

and consequently,

$$\begin{aligned} \Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] &= \Delta\left[-(-\Delta y(k))^{\alpha}\right] \\ &= \sum_{i=1}^{n} p_i(k)[\bar{y}(g_i(k))]^{\alpha} = \sum_{i=1}^{n} p_i(k)\left[y(g_i(k))\right]^{\alpha} \\ &= \sum_{i=1}^{n} p_i(k)|y(g_i(k))|^{\alpha-1}y(g_i(k)), \qquad k \geq K_0. \end{aligned} \tag{3.23}$$

Thus, for sufficiently large k, the $y(k)$ given in (3.22) is a solution of (1.1) satisfying (3.20).

Finally, we shall show that the $y(k)$ given in (3.22) is nonoscillatory. For this, it suffices to prove that $y(k) > 0$ for $k \geq K_0$. Suppose that $y(K_0) = 0$. Since $y(k)$ is nonnegative and decreasing for $k \geq K_0$, it follows that $y(k)$ is identically zero for $k \geq K_0$. Hence, we get

$$\Delta\left[|\Delta y(K_0)|^{\alpha-1}\Delta y(K_0)\right] = 0. \tag{3.24}$$

Now, using the fact that $\bar{y}(k) > 0$, $a \leq k \leq K_0 - 1$ (by definition) and condition (3.18) in (3.23), we find

$$\Delta\left[|\Delta y(k)|^{\alpha-1}\Delta y(k)\right] \geq p_j(k)[\bar{y}(g_j(k))]^{\alpha} > 0, \qquad a \leq k \leq K_0.$$

The above inequality particularly gives

$$\Delta\left[|\Delta y(K_0)|^{\alpha-1}\Delta y(K_0)\right] > 0,$$

which is a contradiction to (3.24). Hence, $y(K_0) > 0$.

Next, let K_1 ($> K_0$) be the first zero of $y(k)$. Then, by definition $\bar{y}(k) > 0$, $a \le k \le K_1 - 1$. Using a similar argument as before (replacing K_0 by K_1), from (3.23) we get

$$\Delta\left[|\Delta y(K_1)|^{\alpha-1}\Delta y(K_1)\right] > 0. \tag{3.25}$$

On the other hand, $y(k)$ is identically zero for $k \ge K_1$. Hence, it follows that

$$\Delta\left[|\Delta y(K_1)|^{\alpha-1}\Delta y(K_1)\right] = 0,$$

which contradicts (3.25). By induction, we see that $y(k) > 0$ for $k \ge K_0$. The proof of the theorem is complete.

THEOREM 3.4. *Suppose that there exists a j, $1 \le j \le n$ such that (3.18) holds. Further, suppose that (3.13) holds. Then, (1.1) has a decaying nonoscillatory solution $\{y(k)\}$ satisfying (3.20).*

PROOF. We shall show that there exists a positive decreasing function $q(k)$ such that (3.19) is satisfied. Then, the result follows immediately from Theorem 3.3. For this, let K_0 ($> a$) be sufficiently large so that

$$\min_{1\le i\le n}\ \min_{k\ge K_0} g_i(k) \ge \max\{a, 1\}, \tag{3.26}$$

and in view of (3.13),

$$\sum_{\ell=K_0+1}^{\infty}\left[\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n} p_i(\tau)\right]^{1/\alpha} \le \frac{1}{2}. \tag{3.27}$$

Let

$$q(k) = 1 + \frac{1}{k}. \tag{3.28}$$

Clearly, $q(k)$ is positive and decreasing. To see that (3.19) holds, for $k \ge K_0$ we have, on using (3.28), (3.26), and (3.27),

$$\begin{aligned}
\sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell}^{\infty}\sum_{i=1}^{n} p_i(\tau-1)[q(g_i(\tau-1))]^{\alpha}\right\}^{1/\alpha} &= \sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n} p_i(\tau)\left[1+\frac{1}{g_i(\tau)}\right]^{\alpha}\right\}^{1/\alpha} \\
&\le 2\sum_{\ell=k+1}^{\infty}\left[\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n} p_i(\tau)\right]^{1/\alpha} \\
&\le 2\sum_{\ell=K_0+1}^{\infty}\left[\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n} p_i(\tau)\right]^{1/\alpha} \le 2\cdot\frac{1}{2} = 1 < q(k).
\end{aligned}$$

This completes the proof of the theorem.

THEOREM 3.5. *Suppose that there exists a j, $1 \le j \le n$ such that (3.18) holds, and*

$$\limsup_{k\to\infty}\sum_{\ell=g_0(k)}^{k}\left[\sum_{\tau=\ell}^{\infty}\sum_{i=1}^{n} p_i(\tau)\right]^{1/\alpha} < \frac{1}{e}, \tag{3.29}$$

where $g_0(k) = \min_{1\le i\le n} g_i(k)$. Then, (1.1) has a decaying nonoscillatory solution $\{y(k)\}$ satisfying (3.20).

PROOF. Again, the result follows immediately from Theorem 3.3, if we can prove that there exists a positive decreasing function $q(k)$ such that (3.19) is satisfied. For this, we denote

$$\phi(\ell) \equiv \left[\sum_{\tau=\ell}^{\infty}\sum_{i=1}^{n} p_i(\tau)\right]^{1/\alpha}.$$

Let K_0 ($> a+1$) be sufficiently large so that

$$\min_{k \geq K_0} g_0(k) \geq a, \tag{3.30}$$

and in view of (3.29),

$$Q \equiv \sup_{k \geq K_0} \sum_{\ell=g_0(k)}^{k} \phi(\ell) \leq \frac{1}{e}. \tag{3.31}$$

We choose

$$q(k) = \exp\left(-\frac{1}{Q}\sum_{\ell=a}^{k}\phi(\ell)\right).$$

It is obvious that $q(k)$ is positive and decreasing. To see that (3.19) holds, we note that for $1 \leq i \leq n$, $k \geq K_0$,

$$\begin{aligned} q(g_i(k)) &= \exp\left(-\frac{1}{Q}\sum_{\ell=a}^{g_i(k)}\phi(\ell)\right) \\ &= \exp\left(\frac{1}{Q}\sum_{\ell=g_i(k)+1}^{k}\phi(\ell)\right)\cdot\exp\left(-\frac{1}{Q}\sum_{\ell=a}^{k}\phi(\ell)\right) \\ &\leq \exp\left(\frac{1}{Q}\,Q\right)\cdot\exp\left(-\frac{1}{Q}\sum_{\ell=a}^{k}\phi(\ell)\right) \\ &= e\,q(k). \end{aligned} \tag{3.32}$$

Using (3.32) and (3.31), for $k \geq K_0$, we find that

$$\begin{aligned} \sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell}^{\infty}\sum_{i=1}^{n}p_i(\tau-1)[q(g_i(\tau-1))]^{\alpha}\right\}^{1/\alpha} &\leq \sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n}p_i(\tau)[e\,q(\tau)]^{\alpha}\right\}^{1/\alpha} \\ &\leq \sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n}p_i(\tau)[e\,q(\ell-1)]^{\alpha}\right\}^{1/\alpha} \\ &= e\sum_{\ell=k}^{\infty}\phi(\ell)q(\ell) \leq e\sum_{\ell=k}^{\infty}\phi(\ell)q(k) \leq eQq(k) \leq q(k), \end{aligned}$$

and hence, (3.19) is satisfied. The proof of the theorem is complete.

THEOREM 3.6. *Suppose that there exists a j, $1 \leq j \leq n$ such that (3.18) holds, and there is a K_0 ($> a+1$) such that (3.30) holds. Further, assume that*

$$\sup_{k \geq K_0} \sum_{\ell=g_0(k)}^{k}\sum_{i=1}^{n}p_i(\ell) < \infty \tag{3.33}$$

and

$$\sum_{\ell=k+1}^{\infty}\left[\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n}p_i(\tau)\right]^{1/\alpha} \leq \exp\left(-\frac{\alpha+1}{\alpha}\right). \tag{3.34}$$

Then, (1.1) has a decaying nonoscillatory solution $\{y(k)\}$ satisfying (3.20).

PROOF. Again, in view of Theorem 3.3, it suffices to prove that there exists a positive decreasing function $q(k)$ such that (3.19) is satisfied. For this, we denote

$$Q \equiv \sup_{k \geq K_0} \sum_{\ell=g_0(k)}^{k}\sum_{i=1}^{n}p_i(\ell) > 0,$$

and let

$$q(k) = \exp\left(-\frac{\alpha+1}{\alpha Q}\sum_{\ell=a}^{k}\sum_{i=1}^{n} p_i(\ell)\right).$$

Clearly, $q(k)$ is positive and decreasing. To see that (3.19) holds, we note that for $1 \le i \le n$, $k \ge K_0$,

$$\begin{aligned} q(g_i(k)) &= \exp\left(-\frac{\alpha+1}{\alpha Q}\sum_{\ell=a}^{g_i(k)}\sum_{i=1}^{n} p_i(\ell)\right) \\ &= \exp\left(\frac{\alpha+1}{\alpha Q}\sum_{\ell=g_i(k)+1}^{k}\sum_{i=1}^{n} p_i(\ell)\right)\cdot \exp\left(-\frac{\alpha+1}{\alpha Q}\sum_{\ell=a}^{k}\sum_{i=1}^{n} p_i(\ell)\right) \\ &\le \exp\left(\frac{\alpha+1}{\alpha Q}Q\right)\cdot \exp\left(-\frac{\alpha+1}{\alpha Q}\sum_{\ell=a}^{k}\sum_{i=1}^{n} p_i(\ell)\right) \\ &= \exp\left(\frac{\alpha+1}{\alpha}\right) q(k). \end{aligned} \tag{3.35}$$

Applying (3.35) and (3.34), for $k \ge K_0$, we find

$$\begin{aligned} \sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell}^{\infty}\sum_{i=1}^{n} p_i(\tau-1)\left[q(g_i(\tau-1))\right]^{\alpha}\right\}^{1/\alpha} &\le \sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n} p_i(\tau)\left[\exp\left(\frac{\alpha+1}{\alpha}\right) q(\tau)\right]^{\alpha}\right\}^{1/\alpha} \\ &\le \sum_{\ell=k+1}^{\infty}\left\{\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n} p_i(\tau)\left[\exp\left(\frac{\alpha+1}{\alpha}\right) q(k)\right]^{\alpha}\right\}^{1/\alpha} \\ &= \exp\left(\frac{\alpha+1}{\alpha}\right) q(k)\sum_{\ell=k+1}^{\infty}\left[\sum_{\tau=\ell-1}^{\infty}\sum_{i=1}^{n} p_i(\tau)\right]^{1/\alpha} \\ &\le \exp\left(\frac{\alpha+1}{\alpha}\right) q(k)\exp\left(-\frac{\alpha+1}{\alpha}\right) = q(k), \end{aligned}$$

and hence, (3.19) is fulfilled. This completes the proof of the theorem.

EXAMPLE 3.1. *Consider the difference equation*

$$\Delta\left[\Delta y(k)\right] = \frac{2}{k(k+2)(k^2+2k+2)} y(k+1), \qquad k \ge 1. \tag{3.36}$$

Here, $\alpha = n = 1$ and condition (3.3) is fulfilled because

$$\sum^{\infty} p_1(k) g_1(k) = \sum^{\infty} \frac{2}{k(k+2)(k^2+2k+2)}(k+1) < \sum^{\infty}\frac{2}{k(k^2+2k+2)} < \infty.$$

Hence, by Theorem 3.1 equation (3.36) has a unbounded nonoscillatory solution $\{y(k)\}$ such that (3.2) is satisfied. In fact, one such solution is given by $\{y(k)\} = \{k + 1/k\}$ and we note that $y(k)/k \to 1$ as $k \to \infty$.

EXAMPLE 3.2. *Consider the difference equation*

$$\Delta\left[|\Delta y(k)|\Delta y(k)\right] = \frac{4}{(k+1)^5}\,|y(k(k+2))|y(k(k+2)), \qquad k \ge 1. \tag{3.37}$$

We have $\alpha = 2$, $n = 1$. Hence,

$$\sum^{\infty}\left[\sum_{\ell=k}^{\infty} p_1(\ell)\right]^{1/\alpha} = \sum^{\infty}\left[\sum_{\ell=k}^{\infty}\frac{4}{(\ell+1)^5}\right]^{1/2} < \sum^{\infty}\left(\int_k^{\infty}\frac{4}{s^5}\,ds\right)^{1/2} = \sum^{\infty}\frac{1}{k^2} < \infty,$$

and so (3.13) is satisfied. It follows from Theorem 3.2 that (3.37) has a bounded nonoscillatory solution $\{y(k)\}$ such that (3.12) holds. In fact, one such solution is given by $\{y(k)\} = \{1 + 1/k\}$ and we note that $y(k) \to 1$ as $k \to \infty$.

REMARK 3.1. *In Example 3.2, we also find*

$$\sum^{\infty} p_1(k)\,[g_1(k)]^2 = \sum^{\infty} \frac{4[k(k+2)]^2}{(k+1)^5} > \sum^{\infty} \frac{4[k(k+1)]^2}{(k+1)^5} = \sum^{\infty} \frac{4k^2}{(k+1)^3} = \infty.$$

Therefore, condition (3.3) is violated. It follows from Theorem 3.1 that equation (3.37) does not have any unbounded nonoscillatory solutions satisfying (3.2).

EXAMPLE 3.3. *Consider the difference equation*

$$\Delta\left[|\Delta y(k)|\Delta y(k)\right] = \frac{4(k+2)^2}{k^2(k+1)}\,\left|y\left((k+2)^2\right)\right| y\left((k+2)^2\right), \qquad k \ge 1. \tag{3.38}$$

We find that

$$\sum^{\infty} p_1(k)[g_1(k)]^2 = \sum^{\infty} \frac{4(k+2)^2}{k^2(k+1)}\,(k+2)^4 = \infty$$

and

$$\sum^{\infty}\left[\sum_{\ell=k}^{\infty} p_1(\ell)\right]^{1/2} = \sum^{\infty}\left[\sum_{\ell=k}^{\infty} \frac{4(\ell+2)^2}{\ell^2(\ell+1)}\right]^{1/2} \ge \sum^{\infty}\left(\sum_{\ell=k}^{\infty} \frac{4}{\ell+1}\right)^{1/2} = \infty.$$

Hence, conditions (3.3) and (3.13) are both violated. By Theorems 3.1 and 3.2, equation (3.38) does not have any unbounded or bounded nonoscillatory solutions satisfying (3.2) and (3.12), respectively. In fact, we note that $\{y(k)\} = \{1/k\}$ is a bounded nonoscillatory solution of (3.38). However, this solution does not fulfill (3.12).

EXAMPLE 3.4. *Consider the difference equation*

$$\Delta\left[|\Delta y(k)|\Delta y(k)\right] = \frac{4[g(k)]^2}{k^2(k+1)(k+2)^2}|y(g(k))|y(g(k)), \qquad k \ge 2, \tag{3.39}$$

where $g(k) = [k + 1/k]$, $[\cdot]$ being the largest integer function.

Clearly, $g(k) < k$ for $k \ge 2$, and hence, condition (3.18) is satisfied. Next, condition (3.13) also holds because

$$\begin{aligned}
\sum^{\infty}\left[\sum_{\ell=k}^{\infty} p_1(\ell)\right]^{1/2} &= \sum^{\infty}\left\{\sum_{\ell=k}^{\infty} \frac{4[g(\ell)]^2}{\ell^2(\ell+1)(\ell+2)^2}\right\}^{1/2} \\
&\le \sum^{\infty}\left[\sum_{\ell=k}^{\infty} 4\left(\frac{\ell+1}{\ell}\right)^2 \frac{1}{\ell^2(\ell+1)(\ell+2)^2}\right]^{1/2} \\
&= 2\sum^{\infty}\left[\sum_{\ell=k}^{\infty} \frac{\ell+1}{\ell^4(\ell+2)^2}\right]^{1/2} \\
&\le 2\sum^{\infty}\left[\sum_{\ell=k}^{\infty} \frac{1}{\ell^5}\right]^{1/2} \\
&\le 2\sum^{\infty}\left(\int_{k-1}^{\infty} \frac{ds}{s^5}\right)^{1/2} = \sum^{\infty} \frac{1}{(k-1)^2} < \infty.
\end{aligned}$$

Hence, it follows from Theorem 3.4 that (3.39) has a decaying nonoscillatory solution $\{y(k)\}$ satisfying (3.20). One such solution is given by $\{y(k)\} = \{1/k\}$.

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